

ERROR ESTIMATE FOR A FULLY DISCRETE SPECTRAL SCHEME FOR KORTEWEG-DE VRIES-KAWAHARA EQUATION.

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ABSTRACT. We are concerned with the convergence of spectral method for the numerical solution of the initial-boundary value problem associated to the Korteweg-de Vries-Kawahara equation (in short Kawahara equation), which is a transport equation perturbed by dispersive terms of 3rd and 5th order. This equation appears in several fluid dynamics problems. It describes the evolution of small but finite amplitude long waves in various problems in fluid dynamics. These equations are discretized in space by the standard Fourier-Galerkin spectral method and in time by the explicit leap-frog scheme. For the resulting fully discrete, conditionally stable scheme we prove an L^2 -error bound of spectral accuracy in space and of second-order accuracy in time.

1. INTRODUCTION

In this paper, we analyze the numerical approximation by Fourier spectral methods to the Korteweg-de Vries-Kawahara (briefly Kawahara) equation with periodic solutions:

$$(1.1) \quad \begin{cases} u_t = -uu_x - u_{xxx} + u_{xxxxx}, & (x, t) \in \mathbb{R} \times [0, \infty), \\ u(x, t) = u(x + 2\pi, t), & (x, t) \in \mathbb{R} \times [0, \infty), \\ u(x, 0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where the initial condition f is a given real valued 2π -periodic function.

It is well known that the one-dimensional waves of small but finite amplitude in dispersive systems (e.g., the magneto-acoustic waves in plasmas, the shallow water waves, the lattice waves and so on) can be described by the Korteweg-de Vries (KdV in short) equation, given by

$$(1.2) \quad u_t = -uu_x - u_{xxx},$$

which admits either compressive or rarefactive steady solitary wave solution (by a solitary water wave, we mean a travelling wave solution of the water wave equations for which the free surface approaches a constant height as $|x| \rightarrow \infty$) according to the sign of the dispersion term (the third order derivative term). Under certain circumstances, however, it might happen that the coefficient of the third order derivative in the KdV equation becomes small or even zero. In that case one has to take account of the higher order effect of dispersion in order to balance the nonlinear effect. In such cases one may obtain a generalized nonlinear dispersive equation, known as Kawahara equation, which has a form of the KdV equation with an additional fifth order derivative term given by (1.1). The Kawahara equation is

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an important nonlinear dispersive equation. It describes solitary wave propagation in media in which the first-order dispersion is anomalously small. A more specific physical background of this equation was introduced by Hunter and Scheurle [8], where they used it to describe the evolution of solitary waves in fluids in which the Bond number is less than but close to $\frac{1}{3}$ and the Froude number is close to 1. In the literature this equation is also referred to as the fifth order KdV equation or singularly perturbed KdV equation. The fifth order term $\partial_x^5 u$ is called the Kawahara term. There has been a great deal of work on solitary wave solutions of the Kawahara equation [11, 12, 9, 16, 6] over the past thirty years. It is found that, similarly to the KdV equation, the Kawahara equation also has solitary wave solutions which decay rapidly to zero as $t \rightarrow \infty$, but unlike the KdV equation whose solitary wave solutions are non-oscillating, the solitary wave solutions of the Kawahara equation have oscillatory trails. This shows that the Kawahara equation is not only similar but also different from the KdV equation in the properties of solutions, like what happens between the formulations of this equation and the KdV equation. The strong physical background of the Kawahara equation and such similarities and differences between it and the KdV equation in both the form and the behavior of the solution render the mathematical treatment of this equation particularly interesting. The Cauchy problem for Kawahara equation has been studied by a few authors [7, 20, 10, 21, 4]. It has been shown that the Cauchy problem has a local solution $u \in C([-T, T]; H^r(\mathbb{R}))$ if $f \in H^r(\mathbb{R})$ and $r > -1$. This local result combined with the energy conservation law yields that (1.1) has a global solution $u \in C([-\infty, \infty]; L^2(\mathbb{R}))$ if $f \in L^2(\mathbb{R})$. Well-posedness results can be found in [7].

Being integrable, Kawahara equation (1.1) has infinitely many invariants. Below we will state only first three of them.

Lemma 1.1. *There exists a unique solution to (1.1). Moreover this solution conserves the first three energy integrals, namely*

$$(1.3) \quad (\partial/\partial t) \left[\int_0^{2\pi} u(x, t) dx \right] = 0,$$

$$(1.4) \quad (\partial/\partial t) \left[\int_0^{2\pi} u^2(x, t) dx \right] = 0,$$

$$(1.5) \quad (\partial/\partial t) \left[\int_0^{2\pi} \left(\frac{1}{3} u^3 - u_x^2 - u_{xx}^2 \right) dx \right] = 0.$$

Proof. The invariance of these expressions can be shown for smooth solutions by using periodicity. For the sake of completeness, we will give a proof.

In order to show (1.3), let us integrate (1.1) in space. We get

$$(\partial/\partial t) \int_0^{2\pi} u dx + (1/2) \int_0^{2\pi} (u^2)_x dx + \int_0^{2\pi} u_{xxx} dx - \int_0^{2\pi} u_{xxxxx} dx = 0,$$

using the periodicity of u , we deduce then (1.3).

To prove (1.4) we start by multiplying the equation (1.1) by u and integrate by parts in space, yields

$$\int_0^{2\pi} uu_t dx = \int_0^{2\pi} -u^2 u_x - uu_{xxx} + uu_{xxxxx} dx$$

$$\begin{aligned}
&= - \int_0^{2\pi} \left(\frac{1}{3}u^3\right)_x dx - \int_0^{2\pi} \left(uu_{xx} - \frac{1}{2}u_x^2\right)_x dx - \int_0^{2\pi} u_x u_{xxxx} dx \\
&= - \int_0^{2\pi} \left(\frac{1}{3}u^3\right)_x dx - \int_0^{2\pi} \left(uu_{xx} - \frac{1}{2}u_x^2\right)_x dx \\
&\quad - \int_0^{2\pi} \left(u_x u_{xxx} - \frac{1}{2}u_{xx}^2\right)_x dx.
\end{aligned}$$

Again using the periodicity of u , we can establish (1.4).

To prove (1.5), we start by multiplying (1.1) by u^2 and integrate by parts in space, yields

$$\begin{aligned}
\int_0^{2\pi} u^2 u_t dx &= \int_0^{2\pi} -u^3 u_x - u^2 u_{xxx} + u^2 u_{xxxx} dx \\
&= - \int_0^{2\pi} \left(\frac{1}{4}u^4\right)_x dx + 2 \int_0^{2\pi} (uu_x)u_{xx} dx - 2 \int_0^{2\pi} (uu_x)u_{xxx} dx \\
&= 2 \int_0^{2\pi} [-u_t - u_{xxx} + u_{xxxx}]u_{xx} dx \\
&\quad - 2 \int_0^{2\pi} [-u_t - u_{xxx} + u_{xxxx}]u_{xxxx} dx \\
&= 2 \int_0^{2\pi} u_{tx}u_x dx - 2 \int_0^{2\pi} u_{xx}u_{xxx} dx + 2 \int_0^{2\pi} u_{xx}u_{xxxx} dx \\
&\quad - 2 \int_0^{2\pi} u_{tx}u_{xxx} dx + 2 \int_0^{2\pi} u_{xxx}u_{xxx} dx - 2 \int_0^{2\pi} u_{xxx}u_{xxxx} dx \\
&= 2 \int_0^{2\pi} u_{tx}u_x dx + 2 \int_0^{2\pi} u_{txx}u_{xx} dx.
\end{aligned}$$

From this we can conclude that

$$\partial/\partial t \left[\int_0^{2\pi} \left(\frac{1}{3}u^3 - u_x^2 - u_{xx}^2 \right) dx \right] = 0.$$

□

There has been a great deal of work on the Fourier-Galerkin spectral method for the KdV equations [1, 2, 14]. Also, spectral methods for initial- and periodic boundary value problems for nonlinear wave equations with nonlocal dispersive terms has been studied by many authors [15]. In this paper, we prove error estimates for a simple spectral fully discrete scheme that we use to approximate spatially periodic solutions of Kawahara equation. We first discretize in space using the standard Fourier-Galerkin spectral method, which is easily shown to preserve the first three invariants of the equation. We prove that the solution of the semi discrete problems converges in L^2 , on bounded temporal intervals, to the solution of the corresponding continuous problem at the spectral rate N^{-r} ; here the number of Fourier points is $2N + 1$ and r is the order of the Sobolev space in which the solution is supposed to belong for $t = 0$.

We then discretize in time the ODE system that results from the spectral semidiscretization using two different methods. First, a second order accurate explicit scheme, the leap frog method and secondly, a second order accurate semi-implicit scheme, the Crank-Nicholson method. We prove the expected $\mathcal{O}(\Delta t^2)$ error bound

in L^2 for both these temporal discretization, for suitably accurate initial conditions and under the stability requirement that $\Delta t N^5$ and $\Delta t N$ respectively are sufficiently small. The same type of mesh restriction is required for stability of any explicit temporal discretization of the stiff ODE semi discrete system under consideration.

The rest of the paper is organized as follows: In section 2, we give all the necessary preliminary results. In section 3, we consider a semi discrete Fourier-Galerkin scheme for the initial-boundary value problem corresponding to (1.1) and prove an error estimate. We prove that the solution of the semi discrete problem converges in L^2 . In section 4, we consider a fully discrete explicit Fourier-Galerkin scheme for the initial-boundary value problem corresponding to (1.1) and prove an error estimate under the stability condition that $\Delta t N^5$ is sufficiently small. Finally, in section 5, we consider a fully discrete semi explicit Fourier-Galerkin scheme for the initial-boundary value problem corresponding to (1.1) and prove an error estimate under the stability condition that $\Delta t N$ is sufficiently small.

2. NOTATION AND PRELIMINARY RESULTS

We consider functions that are periodic of period 2π . The function spaces we use here are L^2 and the Sobolev spaces H^r for integer $r \geq 0$. These spaces will always be considered on $[-\pi, \pi]$ and their elements will be periodic functions. We denote by $(.,.)$ the standard L^2 inner product; this yields a norm in L^2 which we denote by $\|\cdot\|$. The norm in H^r , denoted by $\|\cdot\|_r$, is defined by

$$(2.1) \quad \|f\|_r = \left(\sum_{k \in \mathbb{Z}} (1 + k^2)^r |\hat{f}(k)|^2 \right)^{1/2}.$$

By $\|\cdot\|_\infty$, we denote the norm of $L^\infty = L^\infty[-\pi, \pi]$.

As usual, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$, the Fourier coefficients of f :

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx.$$

We recall that the Fourier coefficients of the pointwise product are given by the convolution of the Fourier coefficients of f and g , defined by

$$(\hat{f} * \hat{g})(k) = \sum_{m, n \in \mathbb{Z}; m+n=k} \hat{f}(m) \hat{g}(n).$$

We also need to consider discrete analogues of the quantities defined above. To this end, for N a positive integer, consider the space S_N defined by

$$(2.2) \quad S_N = \text{span}\{exp(ikx) : k \in \mathbb{Z}, -N \leq k \leq N\}.$$

Let P_N denote the L^2 orthogonal projection onto S_N . The projection has the following approximation properties, whose proof is standard.

Proposition 2.1. *Given integers $0 \leq s \leq r$, there exists a constant C independent of N such that, for any $f \in H^r$,*

$$(2.3) \quad \|f - P_N f\|_s \leq C N^{s-r} \|f\|_r,$$

$$(2.4) \quad \|f - P_N f\|_\infty \leq C N^{1/2-r} \|f\|_r, \quad r \geq 1.$$

Proposition 2.2. *Given integers $0 \leq s \leq r$, there exists a constant C independent of N such that, for any $\psi \in S_N$,*

$$(2.5) \quad \|\psi\|_r \leq CN^{r-s} \|\psi\|_s, \quad \text{and} \quad \|\psi\|_\infty \leq CN^{1/2} \|\psi\|.$$

We use many times in the proofs a Sobolev-type inequality, which states that there exists a constant $C > 0$ such that for all $f \in H^1$

$$(2.6) \quad \|f\|_\infty \leq C \|f\|^{1/2} \|f\|_1^{1/2}.$$

As we have already mentioned in the introduction, we shall consider the initial- and periodic boundary-value problem for the Kawahara equation: We seek a real-valued function $u(x, t)$, 2π periodic in x and satisfying

$$(2.7) \quad \begin{cases} u_t + uu_x + u_{xxx} = u_{xxxxx}, & x \in [-\pi, \pi], \quad t \geq 0, \\ u(x, 0) = f(x), & x \in [-\pi, \pi]. \end{cases}$$

Here $f(x)$ is a real-valued, 2π periodic function.

We now going to state a existence and uniqueness result for the solution of (2.7). See [13] for a proof.

Theorem 2.1. *Let f be in H^r , with $r \geq 5$. Then there exists a unique solution u of (2.7) in H^r , which belongs to the space $C^k([0, \infty); H^{r-5k})$ with $r - 5k \geq -1$, i.e., is such that its temporal derivatives up to order k exist and are continuous and bounded on $[0, \infty)$ with values in H^{r-5k} .*

3. SEMI-DISCRETE APPROXIMATION:

In this section we analyze a Fourier-Galerkin scheme for the discretization of (1.1) in the spatial variable.

The semi discrete Fourier-Galerkin (spectral) approximation to (1.1) is a map \mathbf{U} from $[0, \infty)$ to the real-valued elements of S_N such that, for all $\phi \in S_N$:

$$(3.1) \quad \begin{cases} (\mathbf{U}_t + \mathbf{U}\mathbf{U}_x + \mathbf{U}_{xxx} - \mathbf{U}_{xxxxx}, \phi) = 0, & t \geq 0, \\ \mathbf{U}(0) = P_N f, \end{cases}$$

where P_N is the orthogonal projection of L^2 onto S_N .

By choosing $\phi = e^{ikx}$ for $k = -N, \dots, N$, we see that (3.1) is an initial-value problem for an ODE system for Fourier coefficients $\hat{\mathbf{U}}(k, t)$ of \mathbf{U} . Since \mathbf{U} is real, these coefficients must satisfy the condition $\hat{\mathbf{U}}(k, t) = \overline{\hat{\mathbf{U}}(-k, t)}$ and the equation

$$(3.2) \quad \begin{aligned} \hat{\mathbf{U}}_t(k, t) &= \frac{-ik}{2} \hat{\mathbf{U}} * \hat{\mathbf{U}}(k, t) - k^3 \hat{\mathbf{U}}(k, t) - k^5 \hat{\mathbf{U}}(k, t), \quad -N \leq k \leq N \\ \hat{\mathbf{U}}(k, 0) &= \hat{f}(k). \end{aligned}$$

The right hand side of the system (3.2) is Lipschitz continuous, at least locally, with respect to l^2 norm. Hence, the existence of a maximal time t_0 , $0 < t_0 \leq T$ such that, for all $t < t_0$, there exists a unique solution $U(t)$ to problem (3.1) is a classical result of the theory of differential systems. The problem is to get the existence for an arbitrary time t_0 , or equivalently to prove that one can take $t_0 = T$. This result is a consequence of the fact that (3.1) is conservative in L^2 , which ensures that the solution cannot blow-up.

Now we will present the main properties enjoyed by the Fourier-Galerkin approximation (3.1). More precisely, this semidiscretization preserves the discrete analogues of the first three invariants of (2.7).

Lemma 3.1. *There exists a unique solution \mathbf{U} to problem (3.1). Moreover this solution conserves the first three energy integrals of Kawahara equation, namely*

$$(3.3) \quad (\partial/\partial t) \left[\int_{-\pi}^{\pi} \mathbf{U}(x, t) dx \right] = 0,$$

$$(3.4) \quad (\partial/\partial t) \left[\int_{-\pi}^{\pi} \mathbf{U}^2(x, t) dx \right] = 0,$$

$$(3.5) \quad (\partial/\partial t) \left[\int_{-\pi}^{\pi} \left(\frac{1}{3} \mathbf{U}^3 - \mathbf{U}_x^2 - \mathbf{U}_{xx}^2 \right) (x, t) dx \right] = 0.$$

Proof. First of all, we have already discussed about the existence of a unique solution to (3.1). Now in order to show (3.3), let us first choose $\phi = 1$ as a test function in (3.1). We get

$$\begin{aligned} (\partial/\partial t) \int_{-\pi}^{\pi} \mathbf{U}(x, t) dx + (1/2) \int_{-\pi}^{\pi} \mathbf{U}_x^2(x, t) dx + \int_{-\pi}^{\pi} \mathbf{U}_{xxx}(x, t) dx \\ - \int_{-\pi}^{\pi} \mathbf{U}_{xxxxx}(x, t) dx = 0, \end{aligned}$$

using the periodicity of \mathbf{U} , we deduce then (3.3). To prove (3.4), we choose $\phi = \mathbf{U}$ in (3.1). We obtain

$$\begin{aligned} (\partial/\partial t) \int_{-\pi}^{\pi} \mathbf{U}^2(x, t) dx + (1/3) \int_{-\pi}^{\pi} \mathbf{U}_x^3(x, t) dx + \int_{-\pi}^{\pi} (\mathbf{U} \mathbf{U}_{xxx})(x, t) dx \\ - \int_{-\pi}^{\pi} (\mathbf{U} \mathbf{U}_{xxxxx})(x, t) dx = 0. \end{aligned}$$

Integrating by parts and using the periodicity of \mathbf{U} yields

$$\int_{-\pi}^{\pi} (\mathbf{U} \mathbf{U}_{xxx})(x, t) dx = -(1/2) \int_{-\pi}^{\pi} (\mathbf{U}_x^2)_x(x, t) dx = 0.$$

Similarly, we have

$$\int_{-\pi}^{\pi} (\mathbf{U} \mathbf{U}_{xxxxx})(x, t) dx = 0, \quad \text{and} \quad \int_{-\pi}^{\pi} \mathbf{U}_x^3(x, t) dx = 0.$$

Hence, we deduce (3.4).

We derive now (3.5) by choosing $\phi = P_N(\mathbf{U}^2)$ in (3.1). As a result we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \mathbf{U}_t P_N \mathbf{U}^2 dx + \int_{-\pi}^{\pi} \mathbf{U} \mathbf{U}_x P_N \mathbf{U}^2 dx + \int_{-\pi}^{\pi} \mathbf{U}_{xxx} P_N \mathbf{U}^2 dx \\ - \int_{-\pi}^{\pi} \mathbf{U}_{xxxxx} P_N \mathbf{U}^2 dx = 0. \end{aligned}$$

Now as \mathbf{U}_t is an element of S_N , we have

$$\int_{-\pi}^{\pi} \mathbf{U}_t P_N \mathbf{U}^2 dx = \int_{-\pi}^{\pi} \mathbf{U}_t \mathbf{U}^2 dx = (1/3) (\partial/\partial t) \int_{-\pi}^{\pi} \mathbf{U}^3 dx.$$

On the other hand,

$$\int_{-\pi}^{\pi} \mathbf{U}_{xxx} P_N \mathbf{U}^2 dx = \int_{-\pi}^{\pi} \mathbf{U}_{xxx} \mathbf{U}^2 dx = -2 \int_{-\pi}^{\pi} \mathbf{U}_{xx} (\mathbf{U} \mathbf{U}_x) dx$$

$$= - \int_{-\pi}^{\pi} \mathbf{U}_{xx} (-\mathbf{U}_t - \mathbf{U}_{xxx} + \mathbf{U}_{xxxx}) dx = -(\partial/\partial t) \int_{-\pi}^{\pi} \mathbf{U}_x^2 dx.$$

Similarly we have

$$- \int_{-\pi}^{\pi} \mathbf{U}_{xxxx} P_N \mathbf{U}^2 dx = -(\partial/\partial t) \int_{-\pi}^{\pi} \mathbf{U}_{xx}^2 dx.$$

Finally, using the fact that P_N commutes with differentiation, we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \mathbf{U} \mathbf{U}_x P_N \mathbf{U}^2 dx &= \int_{-\pi}^{\pi} \left(\frac{1}{2} \mathbf{U}^2 \right)_x P_N \mathbf{U}^2 dx \\ &= - \int_{-\pi}^{\pi} \mathbf{U}^2 P_N (\mathbf{U} \mathbf{U}_x) dx = - \int_{-\pi}^{\pi} \mathbf{U} \mathbf{U}_x P_N \mathbf{U}^2 dx, \end{aligned}$$

consequently, we have

$$\int_{-\pi}^{\pi} \mathbf{U} \mathbf{U}_x P_N \mathbf{U}^2 dx = 0.$$

Combining all the results above we get (3.5). \square

We shall now state the following theorem:

Theorem 3.1. *The semi-discrete scheme (3.1) has an unique solution \mathbf{U} for $t \geq 0$. Let $u(x, t)$ be the solution of (1.1) corresponding to the initial data $u_0 \in H^r$. Then there exists a time $T > 0$, and a constant $C > 0$, independent of N , such that*

$$(3.6) \quad \max_{0 \leq t \leq T} \|u - \mathbf{U}\| \leq \frac{C}{N^r}.$$

Before giving the proof of Theorem 3.1, we define a different semi-discretization based on a linearization of (3.1). The convergence of this approximation will serve as an intermediate step in the proof of the convergence of the original scheme.

To this end, we linearize (3.1) as follows: Given a solution of (1.1), corresponding to initial data u_0 in H^r , we look for a function $V \in S_N$, which for all $\phi \in S_N$, satisfies

$$(3.7) \quad \begin{cases} (V_t + uV_x + V_{xxx} - V_{xxxx}, \phi) = 0, & t \geq 0, \\ V(0) = P_N u_0, \end{cases}$$

Lemma 3.2. *Let $u(x, t)$ be a solution of (1.1) corresponding to initial data $u_0 \in H^r$. Then there exists a unique solution V of (3.7) for all $t \geq 0$. Moreover, given $0 \leq t \leq T$, there exists a constant C independent of N such that*

$$(3.8) \quad \max_{0 \leq t \leq T} \|u - V\| \leq \frac{C}{N^r}.$$

Proof. The existence of an unique local solution of (3.7) is again a consequence of standard ODE theory. To see that we have global existence, we resort to a stability result in the L^2 norm. Choosing $\phi = V$ in (3.7), we obtain

$$\frac{1}{2} \frac{d}{dt} \|V\|^2 \leq \|u_x\|_{\infty} \|V\|^2.$$

Sobolev's inequality and the fact that $u_0 \in H^2$ imply that $\max_{t \geq 0} \|u_x\|_{\infty} \leq C$; thus by the Gronwall inequality, there exists C such that $\max_{0 \leq t \leq T} \|V\| \leq C$, and we can extend the local solution to a solution on every bounded interval $[0, T]$.

Note that using the same arguments and choosing $\phi = V_x$, we conclude that $\max_{0 \leq t \leq T} \|V_x\| \leq C$.

Now set $\rho = P_N u - V$, then $u - V = u - P_N u + P_N u - V = u - P_N u + \rho$. The hypothesis on u imply that $\|u - P_N u\| \leq C/N^r$. Thus we only need to estimate ρ . Observe that, ρ is an element of S_N satisfying the equation

$$(\rho_t + \rho_{xxx} - \rho_{xxxx}, \phi) = -(P_N(uu_x), \phi) + (\rho u V_x, \phi).$$

Now observe that,

$$uu_x - uV_x = (u - P_N u)u_x + (P_N u - V)u_x + Vu_x - uV_x$$

Then, the choice $\phi = \rho$ yields,

$$\frac{1}{2} \frac{d}{dt} \|\rho\|^2 \leq \|u_x\|_\infty \|u - P_N u\| \|\rho\| + \|u_x\|_\infty \|\rho\|^2 + \|u_x\|_\infty \|\rho\| \|V\| + \|u\|_\infty \|\rho\| \|V_x\|,$$

so that by the arithmetic-geometric inequality we obtain,

$$\frac{1}{2} \frac{d}{dt} \|\rho\|^2 \leq \frac{C}{(N^r)^2} + C \|\rho\|^2.$$

But since $\rho(x, 0) = 0$, hence we obtain by using Grownwall's inequality

$$\max_{0 \leq t \leq T} \|\rho\| \leq \frac{C}{N^r},$$

and this completes the proof of the lemma. \square

Proof. (of Theorem 3.1)

We have already proved the existence and uniqueness of the semi-discrete solution \mathbf{U} . Now set $e = V - \mathbf{U}$. Then $u - \mathbf{U} = u - V + e$. In view of (3.8), we only need an estimate for e . Now observe that e satisfies

$$(e_t + e_{xxx} - e_{xxxx}, \phi) = -(uV_x - \mathbf{U}\mathbf{U}_x, \phi).$$

Now using the fact that

$$uV_x - \mathbf{U}\mathbf{U}_x = (u - V)V_x + (eV)_x - ee_x$$

and choosing $\phi = e$, observing that $(e, ee_x) = 0$, yields,

$$(3.9) \quad \frac{1}{2} \frac{d}{dt} \|e\|^2 \leq \|V_x\|_\infty \|u - V\| \|e\| + \frac{1}{2} \|V_x\|_\infty \|e\|^2.$$

We now use the following **inverse** inequalities: for $0 \leq s \leq r$ and $\psi \in S_N$

$$\|\psi\|_r \leq CN^{r-s} \|\psi\|_s, \quad \|\psi\|_\infty \leq CN^{1/2} \|\psi\|.$$

Since $(P_N u - V)$ is in S_N ,

$$\|(P_N u - V)_x\|_\infty \leq C \|P_N u - V\|_1^{1/2} \|P_N u - V\|_2^{1/2} \leq CN^{3/2} \|P_N u - V\|.$$

Consequently, we have

$$\begin{aligned} \|P_N u - V\|_\infty &\leq CN^{1/2-r}, \\ \|(P_N u - V)_x\|_\infty &\leq CN^{3/2-r}. \end{aligned}$$

Using Sobolev's inequality along with the approximation properties of the projection P_N , we have

$$\begin{aligned} \|V_x\|_\infty &\leq \|u_x\|_\infty + \|(u - P_N u)_x\|_\infty + \|(P_N u - V)_x\|_\infty \\ &\leq C + C \|u - P_N u\|_2 + \frac{C}{N^{r-3/2}} \end{aligned}$$

$$\leq C + \frac{C}{N^{r-2}} + \frac{C}{N^{r-3/2}}.$$

Using $r \geq 2$, these inequalities yield

$$\max_{0 \leq t \leq T} \|V_x\|_\infty \leq C,$$

consequently (using arithmetic-geometric inequality), from (3.9)

$$\frac{1}{2} \frac{d}{dt} \|e\|^2 \leq C \left(\|e\|^2 + \|u - V\|^2 \right).$$

Finally, writing $u - V = u - P_N u + \rho$, we have

$$\frac{1}{2} \frac{d}{dt} \|e\|^2 \leq C \|e\|^2 + \left(\frac{C}{N^r} \right)^2.$$

Since $e(0) = 0$, Gronwall's inequality gives,

$$\max_{0 \leq t \leq T} \|e\| \leq \frac{C}{N^r},$$

which in view of (3.8), yields (3.6). \square

4. FULLY-DISCRETE SCHEME:

To define the fully discrete scheme, given $0 < T < \infty$, choose a time step Δt , and an integer M , such that $M\Delta t = T$. Then for $m = 0, \dots, M$, denote $t_m = m\Delta t$. The fully discrete solution is defined as the sequence $\{\mathbf{U}^m\}$ of elements of S_N satisfying, for all $\phi \in S_N$ and for $m = 1, 2, \dots, M$, the equation

$$(4.1) \quad (\mathbf{U}^{m+1} - \mathbf{U}^{m-1}, \phi) + 2\Delta t (\mathbf{U}^m \mathbf{U}_x^m + \mathbf{U}_{xxx}^m - \mathbf{U}_{xxxx}^m, \phi) = 0.$$

For each m , \mathbf{U}^m is an approximation of $\mathbf{U}(t_m)$, the semi-discrete solution \mathbf{U} evaluated at time $t = t_m$. We also suppose that initial values $\mathbf{U}^0, \mathbf{U}^1$ have been given in S_N .

Theorem 4.1. *Let $\mathbf{U}(t)$ be the solution of the semi-discrete problem (3.1) and $\{\mathbf{U}^m\}$ be the solution of (4.1). Suppose that $\mathbf{U}^0 = \mathbf{U}(0)$ and that \mathbf{U}^1 is computed in such a way that*

$$(4.2) \quad \|\mathbf{U}^1 - \mathbf{U}(\Delta t)\| \leq C\Delta t^2.$$

Assume that u_0 is in H^r with $r \geq 16$. Then, there exists a constant C_1 independent of N and Δt , such that if

$$(4.3) \quad N^5 \Delta t \leq C_1,$$

there holds

$$(4.4) \quad \max_{0 \leq m \leq M} \|\mathbf{U}^m - \mathbf{U}(t_m)\| \leq C\Delta t^2$$

Proof. We see that $\{\mathbf{U}^m\}$ satisfies for all $\phi \in S_N$:

$$(4.5) \quad (\mathbf{U}^{m+1} - \mathbf{U}^{m-1}, \phi) + 2\Delta t (\mathbf{U}^m \mathbf{U}_x^m + \mathbf{U}_{xxx}^m - \mathbf{U}_{xxxx}^m, \phi) = 0.$$

On the other hand, the semi-discrete solution \mathbf{U} satisfies, for all $\phi \in S_N$:

$$(4.6) \quad (\mathbf{U}(t_{m+1}) - \mathbf{U}(t_{m-1}), \phi) + 2\Delta t (\mathbf{U}(t_m) \mathbf{U}_x(t_m) + \mathbf{U}_{xxx}(t_m) - \mathbf{U}_{xxxx}(t_m), \phi) \\ (4.7) \quad = (\theta^m, \phi),$$

where θ^m is an element of S_N given by:

$$\theta^m = \mathbf{U}(t_{m+1}) - \mathbf{U}(t_{m-1}) - 2\Delta t \mathbf{U}_t(t_m).$$

From Taylor's expansion, we have

$$\|\theta^m\| \leq C\Delta t^3 \max_{t_{m-1} \leq s \leq t_{m+1}} \left\| \frac{\partial^3 \mathbf{U}(s)}{\partial t^3} \right\|.$$

Now let us define $e^m \in S_N$ as

$$e^m = \mathbf{U}^m - \mathbf{U}(t_m).$$

Then e^m satisfies, for all $\phi \in S_N$

$$(4.8) \quad (e^{m+1} - e^{m-1}, \phi) + 2\Delta t (\mathbf{U}^m \mathbf{U}_x^m - \mathbf{U}(t_m) \mathbf{U}_x(t_m) + e_{xxx}^m - e_{xxxx}^m, \phi) = -(\theta^m, \phi).$$

Choosing $\phi = e^{m+1} + e^{m-1}$, and adding $\|e^m\|^2$ to both sides of (4.8), we obtain

$$(4.9) \quad \begin{aligned} & \|e^{m+1}\|^2 + \|e^m\|^2 - 2\Delta t (e_{xx}^m, e_x^{m+1}) - 2\Delta t (e_{xxx}^m, e_{xx}^{m+1}) \\ &= \|e^m\|^2 + \|e^{m-1}\|^2 - 2\Delta t (e_{xx}^{m-1}, e_x^m) - 2\Delta t (e_{xxx}^{m-1}, e_{xx}^m) - (\theta^m, e^{m+1} + e^{m-1}) \\ & \quad - 2\Delta t (\mathbf{U}_x^m e^m, e^{m+1} + e^{m-1}) - 2\Delta t (\mathbf{U}(t_m) e_x^m, e^{m+1} + e^{m-1}), \end{aligned}$$

where we have used the following identity

$$\mathbf{U}^m \mathbf{U}_x^m - \mathbf{U}(t_m) \mathbf{U}_x(t_m) = \mathbf{U}_x^m e^m + \mathbf{U}(t_m) e_x^m.$$

Now let us define A^{m+1} by

$$(4.10) \quad A^{m+1} = \|e^{m+1}\|^2 + \|e^m\|^2 - 2\Delta t (e_{xx}^m, e_x^{m+1}) - 2\Delta t (e_{xxx}^m, e_{xx}^{m+1})$$

$$(4.11) \quad - 2\Delta t (\mathbf{U}(t_m) e_x^{m+1}, e^m).$$

Then we can rewrite (4.9) as

$$(4.12) \quad \begin{aligned} A^{m+1} &= A^m - 2\Delta t (\mathbf{U}_x^m e^m, e^{m+1} + e^{m-1}) + 2\Delta t (\mathbf{U}_x(t_m) e^m, e^{m+1}) \\ & \quad - 2\Delta t ((\mathbf{U}(t_m) - \mathbf{U}(t_{m-1})) e_x^m, e^{m-1}) - (\theta^m, e^{m+1} + e^{m-1}). \end{aligned}$$

In general, differentiating (3.1) with respect to t and using the properties of P_N , it is straightforward to prove that there exist constants $\alpha_{k,s}$, independent of N , such that, if $r \geq s + 5k + 1$

$$\max_{0 \leq t \leq T} \|\partial_t^k \mathbf{U}(t)\|_s \leq \alpha_{k,s}.$$

In particular, since we assume $r \geq 16$, we have by Sobolev's inequality

$$\|\mathbf{U}(t_m) - \mathbf{U}(t_{m-1})\| \leq \Delta t \max_{t_{m-1} \leq s \leq t_m} \|\mathbf{U}_t(s)\|_\infty \leq C\Delta t,$$

and

$$\|\theta^m\| \leq C\Delta t^3.$$

Again, (4.12) gives after some manipulations:

$$(4.13) \quad \begin{aligned} A^{m+1} &\leq A^m + C\Delta t^2 N \|e^m\| \|e^{m+1} + e^{m-1}\| + C\Delta t \|\mathbf{U}_x^m\|_\infty \|e^m\| \|e^{m+1} + e^{m-1}\| \\ & \quad + C\Delta t \|e^m\| \|e^{m+1}\| + C\Delta t^3 \|e^{m+1} + e^{m-1}\|. \end{aligned}$$

Now as an internal “inductive” hypothesis, we assume that there exists a constant B , independent of N , such that for all $n \leq m$

$$\|\mathbf{U}_x^m\|_\infty \leq B,$$

then using the Cauchy-Schwartz inequality, we have

$$(4.14) \quad A^{m+1} \leq A^m + C\Delta t^5 + C\Delta t(1 + B + N\Delta t) \left(\|e^{m-1}\|^2 + 2\|e^m\|^2 + \|e^{m+1}\|^2 \right).$$

On the other hand, one can show that under a stability assumption (4.3), A^{m+1} is positive and comparable to $\|e^m\|^2 + \|e^{m+1}\|^2$. Infact,

$$\begin{aligned} \Delta t |(e_{xx}^m, e_x^{m+1}) + (e_{xxx}^m, e_{xx}^{m+1}) + (\mathbf{U}(t_m)e_x^{m+1}, e^m)| \\ \leq C\Delta t(N^3 + N^5) \left(\|e^m\|^2 + \|e^{m+1}\|^2 \right). \end{aligned}$$

Hence, if

$$C\Delta t(N^3 + N^5) \leq \frac{1}{2},$$

i.e., under a condition of the form (4.3), we have

$$\frac{1}{2} \left(\|e^m\|^2 + \|e^{m+1}\|^2 \right) \leq A^{m+1} \leq 2 \left(\|e^m\|^2 + \|e^{m+1}\|^2 \right).$$

Consequently by (4.14), we have

$$A^{m+1} \leq A^m + C\Delta t^5 + C\Delta t(C + B)(A^m + A^{m+1}),$$

which implies, for $C^* = C(C + B)$

$$(1 - C^*\Delta t)A^{m+1} \leq (1 + C^*\Delta t)A^m + C\Delta t^5.$$

Hence, if Δt is chosen small enough, and using the fact that $\|e^1\| = \mathcal{O}(\Delta t^2)$ by (4.2), we obtain in the standard way

$$\max_{1 \leq n \leq m+1} \|e^n\| \leq C\Delta t^2 e^{CT}.$$

Observe that, the above estimate allows us to complete the inductive step. Indeed, we see that

$$\|\mathbf{U}_x^{m+1}\|_\infty \leq \|\mathbf{U}_x(t_{m+1})\|_\infty + \|e_x^{m+1}\|_\infty \leq \|\mathbf{U}_x(t_{m+1})\|_\infty + C\Delta t^2 N^{3/2} e^{CT} \leq B$$

consequently, by taking Δt sufficiently small and using a condition of type (4.3), we can justify the assumption that $\|\mathbf{U}_x^m\|_\infty \leq B$ for all m . \square

Remark 4.1. *In conclusion, combining the results of Theorems 3.1 and 4.1, we see that under the hypotheses of Theorem 4.1, the fully discrete scheme (4.1) satisfies the error estimate*

$$\max_{0 \leq m \leq M} \|u(t_m) - \mathbf{U}^m\| \leq C \left(\frac{1}{N^r} + \Delta t^2 \right).$$

5. SEMI-IMPLICIT SCHEME

To define the semi-implicit fully discrete scheme, given $0 < T < \infty$, choose a time step Δt , and an integer M , such that $M\Delta t = T$. Then for $m = 0, \dots, M$, denote $t_m = m\Delta t$. The fully discrete solution is defined as the sequence $\{\mathbf{U}^m\}$ of elements of S_N satisfying, for all $\phi \in S_N$ and for $m = 1, 2, \dots, M$, the equation

$$(5.1) \quad (\mathbf{U}^{m+1} - \mathbf{U}^m, \phi) + \Delta t \left(\mathbf{U}^{m+1/2} \mathbf{U}_x^{m+1/2} + \mathbf{U}_{xxx}^{m+1/2} - \mathbf{U}_{xxxx}^{m+1/2}, \phi \right) = 0.$$

For each m , \mathbf{U}^m is an approximation of $\mathbf{U}(t_m)$, the semi-discrete solution \mathbf{U} evaluated at time $t = t_m$. Also, we have used the notation: $\mathbf{U}^{m+1/2} = \frac{1}{2}(\mathbf{U}^m + \mathbf{U}^{m+1})$.

First we shall establish the rate of convergence estimates. So, for the time being, we assume the existence of a sequence $\{\mathbf{U}^n\}_{n=0}^M$ in S_N satisfying (5.1). Later in this section we will discuss about the existence and uniqueness of such sequences.

Theorem 5.1. *Let $\mathbf{U}(t)$ be the solution of the semi-discrete problem (3.1) and $\{\mathbf{U}^m\}$ be the solution of (4.1). Assume that u_0 is in H^r with $r \geq 16$. Then, there exists a constant C_1 independent of N and Δt , such that if*

$$(5.2) \quad N\Delta t \leq C_1,$$

there holds

$$(5.3) \quad \max_{0 \leq m \leq M} \|\mathbf{U}^m - \mathbf{U}(t_m)\| \leq C\Delta t^2$$

Proof. First note that, by letting $\phi = \mathbf{U}^{m+1/2}$ in (5.1), we have by periodicity

$$\begin{aligned} \frac{1}{2} \left(\|\mathbf{U}^{m+1}\|^2 - \|\mathbf{U}^m\|^2 \right) &= - \left(\mathbf{U}^{m+1/2} \mathbf{U}_x^{m+1/2} + \mathbf{U}_{xxx}^{m+1/2} - \mathbf{U}_{xxxx}^{m+1/2}, \mathbf{U}^{m+1/2} \right) \\ &= 0, \end{aligned}$$

hence

$$(5.4) \quad \|\mathbf{U}^m\| = \|\mathbf{U}^0\|, \quad m = 0, \dots, M.$$

We see that $\{\mathbf{U}^m\}$ satisfies for all $\phi \in S_N$:

$$(5.5) \quad (\mathbf{U}^{m+1} - \mathbf{U}^m, \phi) + \Delta t \left(\mathbf{U}^{m+1/2} \mathbf{U}_x^{m+1/2} + \mathbf{U}_{xxx}^{m+1/2} - \mathbf{U}_{xxxx}^{m+1/2}, \phi \right) = 0.$$

On the other hand, the semi-discrete solution \mathbf{U} satisfies, for all $\phi \in S_N$:

$$(5.6) \quad \begin{aligned} &(\mathbf{U}(t_{m+1}) - \mathbf{U}(t_m), \phi) \\ &+ \Delta t \left(\mathbf{U}(t_{m+1/2}) \mathbf{U}_x(t_{m+1/2}) + \mathbf{U}_{xxx}(t_{m+1/2}) - \mathbf{U}_{xxxx}(t_{m+1/2}), \phi \right) = (\theta^m, \phi), \end{aligned}$$

where θ^m is an element of S_N given by:

$$\theta^m = \mathbf{U}(t_{m+1}) - \mathbf{U}(t_m) - 2\Delta t \mathbf{U}_t(t_{m+1/2}).$$

From Taylor's expansion, we have

$$\|\theta^m\| \leq C\Delta t^3 \max_{t_{m-1} \leq s \leq t_{m+1}} \left\| \frac{\partial^3 \mathbf{U}(s)}{\partial t^3} \right\|.$$

Now let us define $e^m \in S_N$ as

$$e^m = \mathbf{U}^m - \mathbf{U}(t_m).$$

Then e^m satisfies, for all $\phi \in S_N$

$$(5.7) \quad \begin{aligned} & (e^{m+1} - e^m, \phi) \\ & + \Delta t \left(\mathbf{U}^{m+1/2} \mathbf{U}_x^{m+1/2} - \mathbf{U}(t_{m+1/2}) \mathbf{U}_x(t_{m+1/2}) + e_{xxx}^{m+1/2} - e_{xxxx}^{m+1/2}, \phi \right) = -(\theta^m, \phi). \end{aligned}$$

Now choose $\phi = e^{m+1/2}$ in (5.7) and observe the following estimates:

$$(e^{m+1} - e^m, e^{m+1/2}) = \frac{1}{2} (\|e^{m+1}\|^2 - \|e^m\|^2).$$

On the other hand, we have

$$\begin{aligned} & (\mathbf{U}^{m+1/2} \mathbf{U}_x^{m+1/2} - \mathbf{U}(t_{m+1/2}) \mathbf{U}_x(t_{m+1/2}), e^{m+1/2}) \\ & = (e^{m+1/2} e_x^{m+1/2} + \mathbf{U}(t_{m+1/2}) e_x^{m+1/2} + \mathbf{U}_x(t_{m+1/2}) e^{m+1/2}, e^{m+1/2}), \end{aligned}$$

and finally, using periodicity we can conclude that

$$(e_{xxx}^{m+1/2} - e_{xxxx}^{m+1/2}, e^{m+1/2}) = 0.$$

Keeping all the above estimates in mind, we have the following:

$$\begin{aligned} \frac{1}{2} (\|e^{m+1}\|^2 - \|e^m\|^2) & \leq C \Delta t \|\mathbf{U}_x(t_{m+1/2})\|_\infty \|e^{m+1/2}\|^2 + C \Delta t^3 \|e^{m+1/2}\|^2 \\ & \leq C \Delta t^5 + C \Delta t \|e^{m+1/2}\|^2. \end{aligned}$$

Using a standard argument, we conclude that (5.3) holds. \square

We now turn into the proof of existence of a sequence $\{\mathbf{U}^n\}_{n=0}^M$ satisfying (5.1). For this we shall use the following variant of the well-known fixed point theorem of Brouwer [19].

Lemma 5.1. *Let H be a finite-dimensional Hilbert space with inner product $(\cdot, \cdot)_H$, and norm $\|\cdot\|_H$. Let the map $f : H \rightarrow H$ be continuous. Suppose there exists $\beta > 0$ such that $(f(K), K)_H > 0$ for all K with $\|K\| = \beta$. Then there exists $K^* \in H$, $\|K^*\| \leq \beta$ such that $f(K^*) = 0$.*

The argument of existence of $\{\mathbf{U}^n\}_{n=0}^M$ proceeds in an inductive way. Assume that $\{\mathbf{U}^j\}_{j=0}^n$ exists.

For $K \in S_N$, define $f : S_N \rightarrow S_N$ by

$$(5.8) \quad (f(K), \phi) = (K - 2\mathbf{U}^m, \phi) + \frac{\Delta t}{4} (KK_x, \phi) + \frac{\Delta t}{2} (K_{xxx} - K_{xxxx}, \phi), \quad \forall \phi \in S_N.$$

Such a map exists by the Riesz representation theorem; the fact that f is continuous follows easily from inverse inequalities. Furthermore, by periodicity, letting $\phi = K$

$$(f(K), K) = (K - 2\mathbf{U}^m, K) \geq \|K\| (\|K\| - 2\|\mathbf{U}^m\|) \geq \|K\| (\|K\| - 2\|\mathbf{U}^0\|),$$

from (5.4). Letting $\beta > 2\|\mathbf{U}^0\|$, we deduce the existence via Lemma 5.1 of a $K^* \in S_N$ such that $f(K^*) = 0$. Now letting $\mathbf{U}^{m+1} = K^* - \mathbf{U}^m$, we get from (5.8) that

$$(\mathbf{U}^{m+1} - \mathbf{U}^m, \phi) + \Delta t (\mathbf{U}^{m+1/2} \mathbf{U}_x^{m+1/2} + \mathbf{U}_{xxx}^{m+1/2} - \mathbf{U}_{xxxx}^{m+1/2}, \phi) = 0, \quad \forall \phi \in S_N,$$

proving the existence of \mathbf{U}^{m+1} .

For uniqueness, suppose that $\mathbf{V}^{m+1} \in S_N$ satisfies

$$(5.9) \quad (\mathbf{V}^{m+1} - \mathbf{V}^m, \phi) + \Delta t \left(\mathbf{V}^{m+1/2} \mathbf{V}_x^{m+1/2} + \mathbf{V}_{xxx}^{m+1/2} - \mathbf{V}_{xxxx}^{m+1/2}, \phi \right) = 0.$$

Then using $E^m = \mathbf{U}^m - \mathbf{V}^m$, from (5.1) and (5.9) we have for all $\phi \in S_N$,

$$\begin{aligned} (E^{m+1} - E^m, \phi) \\ = -\Delta t \left(\mathbf{V}^{m+1/2} E_x^{m+1/2} + \mathbf{U}_x^{m+1/2} E^{m+1/2} + E_{xxx}^{m+1/2} - E_{xxxx}^{m+1/2}, \phi \right). \end{aligned}$$

We claim that as long as U^m exists, we have

$$(5.10) \quad \|\mathbf{U}^m\|_\infty \leq C \quad \text{for all } m$$

$$(5.11) \quad \|\mathbf{U}_x^m\|_\infty \leq C \quad \text{for all } m$$

Infact, Theorem 5.1 and some Sobolev inequalities yields,

$$\|U^m - u(t_m)\|_\infty \leq CN^{1/2} \|U^m - u(t_m)\| \leq CN^{1/2} \left(\frac{1}{N^r} + \Delta t^2 \right)$$

$$\|U_x^m - u_x(t_m)\|_\infty \leq CN^{1/2} \|(U^m - u(t_m))_x\| \leq CN^{3/2} \left(\frac{1}{N^r} + \Delta t^2 \right).$$

Using the facts $r \geq 2$, and (5.2), we deduce (5.10) and (5.11).

Letting $\phi = E^{m+1/2}$, we deduce that

$$\begin{aligned} \frac{1}{2} \left(\|E^{m+1}\|^2 - \|E^m\|^2 \right) &\leq C\Delta t \left\| \mathbf{U}_x^{m+1/2} \right\|_\infty \left\| E^{m+1/2} \right\|^2 \\ &\quad + C\Delta t \left\| \mathbf{V}_x^{m+1/2} \right\|_\infty \left\| E^{m+1/2} \right\|^2, \end{aligned}$$

from the above relation, we conclude that

$$\|E^{m+1}\|^2 \leq \nu \|E^m\|^2, \quad \text{with } \nu \leq 1.$$

Now taking $\mathbf{U}^m = \mathbf{V}^m$, we see that $E^{m+1} = 0$, hence uniqueness follows.

Remark 5.1. In conclusion, combining the results of Theorems 3.1 and 5.1, we see that under the hypotheses of Theorem 5.1, the fully discrete scheme (5.1) satisfies the error estimate

$$\max_{0 \leq m \leq M} \|u(t_m) - \mathbf{U}^m\| \leq C \left(\frac{1}{N^r} + \Delta t^2 \right).$$

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